

# Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity

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## Abstract

Strong existence and pathwise uniqueness of solutions with  $L^\infty$ -vorticity of 2D stochastic Euler equations is proved. The noise is multiplicative and involves first derivatives. A Lagrangian approach is implemented, where a stochastic flow solving a nonlinear flow equation is constructed. Stability under regularization is also proved.

## 1 Introduction

The aim of this paper is to prove strong existence and pathwise uniqueness of solutions in  $L^\infty$  for the stochastic 2D Euler equation in vorticity form

$$d\xi + u^\xi \cdot \nabla \xi dt + \sum_{k=1}^{\infty} \sigma_k \cdot \nabla \xi \circ dW^k = 0, \quad \xi|_{t=0} = \xi_0, \quad (1.1)$$

where the initial vorticity  $\xi_0$  belongs to  $L^\infty$  space. The equation above is subject to the periodic boundary conditions and thus can be reformulated as a problem on a 2-dimensional torus  $\mathbb{T}^2 = (\mathbb{R}/(2\pi\mathbb{Z}))^2$ , see for instance [28, chapter 2]. Thus the space variable is assumed to be an element of  $\mathbb{T}^2$  and all fields are assumed to be  $(2\pi)^2$ -periodic (or simply defined on  $\mathbb{T}^2$ ). The noise coefficients  $\sigma_k$ 's are bounded, regular enough, divergence-free vector fields,  $W^k$ 's are independent Brownian motions and the velocity field  $u^\xi$  is defined as

$$u_t^\xi(x) = K * \xi_t(x) = \int_{\mathbb{T}^2} K(x-y) \xi_t(y) dy,$$

where  $K = \nabla^\perp G = (-\partial_2 G, \partial_1 G)$  and  $G$  is the Green function of the Laplacian on the torus; in other words,

$$u^\xi = -\nabla^\perp (-\Delta)^{-1} \xi.$$

We will also prove stability of the solution under regularization of the kernel  $K$ .

The Stratonovich form is the natural one for several reasons, including physical intuition related to Wong-Zakai principle and the fact that an Itô term of the form  $\sum_{k=1}^{\infty} \sigma_k \cdot \nabla \xi dW^k$  would require a compensating second order operator to hope for a well defined system (see [26]). However, for the opportunity of mathematical analysis, we shall formally rewrite the equation in the Itô form

$$\begin{aligned} d\xi + u^\xi \cdot \nabla \xi dt &+ \sum_{k=1}^{\infty} \sigma_k \cdot \nabla \xi dW^k - \frac{1}{2} \sum_{k=1}^{\infty} (\sigma_k \cdot \nabla) \sigma_k \cdot \nabla \xi dt \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \text{tr}[\sigma_k \sigma_k^* D^2 \xi] dt, \end{aligned} \tag{1.2}$$

and give a rigorous interpretation of the latter one (under some simplified assumptions). Nonetheless it is useful to think heuristically in form of the Stratonovich expression sometimes and it would be misleading to believe that the equation has a parabolic character due to the term  $\text{tr}(a D^2 \xi)$  in the Itô formulation.

Multiplicative noise of type of equation (1.1) is used for instance in models of passive transport, see for instance [15] and references therein, and has been proved to have regularizing effects on some PDE, see for instance [13]. See also [23] for a Lagrangian motivation in modelling turbulent fluids.

The literature on stochastic Euler equations counts a number of works, including [5], [6], [7], [8], [9], [10], [14], [17], [18], [24], [25]. When the noise is additive, the theory is easier and more complete, since the equation may be studied pathwise. This allows one to prove the uniqueness in  $L^\infty$  when  $\xi_0 \in L^\infty$ , one of the most demanding results in the stochastic case due to the difficulty of stochastic calculus in  $L^\infty$ ; see also [16] for delicate  $L^\infty$  estimates on invariant measures for stochastic Navier-Stokes equations with additive noise and their inviscid limit. Recall that the uniqueness for  $L^\infty$ -vorticity in the deterministic case is a celebrated result of [29]. When the noise, on the contrary is multiplicative, uniqueness for  $L^\infty$  initial vorticity was an open problem, as well as the existence of an  $L^\infty$ -solution. Other results of existence, in particular of martingale solutions, were known from the works quoted above.

We solve here the problem in  $L^\infty$  following the Lagrangian approach of [22]. It is based, in the stochastic case, on the investigation of the stochastic flow

equation

$$\begin{aligned}\Phi_t(x) &= x + \int_0^t \int_{\mathbb{T}^2} K(\Phi_s(x) - \Phi_s(y)) \xi_0(y) dy \\ &\quad + \sum_k \int_0^t \sigma_k(\Phi_s(x)) dW_s^k,\end{aligned}$$

which is a problem of interest in itself, even when the kernel  $K$  is smooth. This equation is not trivial because of the global dependence of  $\Phi_t(x)$  on  $(\Phi_s(y))_{y \in \mathbb{T}^2}$  and the difficulty to develop stochastic calculus (for instance a fixed point argument) in the space of (measure preserving, continuous) maps  $\psi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ . The approach inspired by [22] allows us to study this equation and apply the result to the existence and uniqueness of equation (1.1) in  $L^\infty$ .

## 2 The main results

Before giving the results, we state the hypotheses with some preliminary remarks.

**Condition 2.1.** *In the paper, we will always assume that  $\xi_0$ , the initial vorticity, belongs to  $L^\infty(\mathbb{T}^2)$ .*

**Condition 2.2.** *The vector fields  $\sigma_k$ 's are divergence-free and in  $C^{0,1}(\mathbb{T}^2)$  (Lipschitz periodic functions); moreover the family  $(\sigma_k)_k$  is in  $\ell^2(C^{0,1})$ , that is*

$$\sum_{k=1}^{\infty} \sup_{x \in \mathbb{T}^2} |\sigma_k(x)|^2 + \sum_{k=1}^{\infty} \sup_{x, y \in \mathbb{T}^2, x \neq y} \frac{|\sigma_k(x) - \sigma_k(y)|^2}{|x - y|^2} < +\infty.$$

We call  $a(x) := \sum_{k=1}^{\infty} \sigma_k(x) \sigma_k(x)^*$  ( $A^*$  denotes the transpose matrix of  $A$ ). We assume also that  $a \equiv C I_2$ , where  $C$  is a non-negative constant (possibly equal to 0) and  $I_2$  is the constant identity matrix.

**Remark 2.3.** *Under hypothesis  $a(x) \equiv I_2$ , the Itô formulation (1.2) simplifies to*

$$d\xi + u^\xi \cdot \nabla \xi dt + \sum_{k=1}^{\infty} \sigma_k \cdot \nabla \xi dW^k = \frac{1}{2} C \Delta \xi dt. \quad (2.1)$$

Indeed, the first order Itô correction term, namely  $\frac{1}{2} \sum_k (\sigma_k \cdot \nabla) \sigma_k \cdot \nabla \xi dt$ , disappears: since the  $\sigma_k$ 's are divergence-free,

$$\sum_k \sum_i \sigma_{k,i}(x) \partial_i \sigma_{k,j} = \sum_i \partial_i \left( \sum_k \sigma_{k,i}(x) \sigma_{k,j}(x) \right) = \sum_i \partial_i a_{ij}(x) = 0.$$

**Remark 2.4.** Condition  $a(x) \equiv I_2$  can be avoided at the price of asking more regularity on the  $\sigma_k$ 's and of a few additional computations, which would obscure the main arguments. Indeed, the fact that  $a$  is constant implies the absence of the first order Itô correction term, which contains the derivatives of  $\sigma$ , and that the operator  $\frac{1}{2}\text{tr}[aD^2] = \frac{1}{2}C\Delta$  commutes with the convolution with a given function; this will avoid the use of a second order commutator lemma (not difficult but boring and requiring maybe more regularity on the  $\sigma_k$ 's).

**Remark 2.5.** Condition  $\sigma \in \ell^2(C^{0,1})$  is slightly stronger than the usual one for flows of homeomorphisms, namely  $\sigma \in C^{0,1}(\ell^2)$ . It will be needed only for uniqueness of the stochastic Euler vorticity equation (not for existence and uniqueness of the associated flow).

**Condition 2.6.** The family of processes  $W = (W^k)_{k \in \mathbb{N}^+}$  is a cylindrical Brownian motion (i.e.  $W^k$ 's are independent Brownian motions), defined on a probability space  $(\Omega, \mathcal{A}, P)$ , with respect to the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ .

**Definition 2.7.** Let  $\xi$  be an element of  $L^\infty([0, T] \times \mathbb{T}^2 \times \Omega)$ . We say that  $\xi$  is weakly progressively measurable (with respect to  $\mathbb{F}$ ) if, for every  $f$  in  $L^1(\mathbb{T}^2)$ , the process  $t \rightarrow \langle \xi_t, f \rangle = \int_{\mathbb{T}^2} f \xi_t dx$  is progressively measurable.

Given an element  $w$  in  $L^\infty(\mathbb{T}^2)$ , we will call

$$u = u^w = K * w.$$

If  $w$  is also time-dependent, we will call  $u_t^w = u^{w_t}$ . It is well known (see Corollary 2.15) that  $|u^w(x) - u^w(y)| \leq \|w\|_{L^\infty} |x - y| (1 - \log |x - y|)$  if  $|x - y| \leq 1$ .

Now we give a precise definition of a solution. We use the Itô formulation, having in mind Remark 2.3. In what follows,  $\langle f, g \rangle := \int_{\mathbb{T}^2} f g dx$  denotes the scalar product in  $L^2(\mathbb{T}^2)$ .

**Definition 2.8.** Let  $\xi_0$  be in  $L^\infty(\mathbb{T}^2)$ . A distributional  $L^\infty$  solution to the stochastic Euler vorticity equation (2.1) is an element  $\xi$  in  $L^\infty([0, T] \times \mathbb{T}^2 \times \Omega)$ , with  $\xi$  and  $u^\xi \xi$  being weakly progressively measurable with respect to  $\mathbb{F}$ , such that, for every  $\varphi$  in  $C^\infty(\mathbb{T}^2)$ , it holds

$$\begin{aligned} \langle \xi_t, \varphi \rangle &= \langle \xi_0, \varphi \rangle + \int_0^t \langle \xi_r, u_r^\xi \cdot \nabla \varphi \rangle dr + \sum_k \int_0^t \langle \xi_r, \sigma_k \cdot \nabla \varphi \rangle dW_r \\ &+ \frac{1}{2} \int_0^t \langle \xi_r, \text{tr}[aD^2 \varphi] \rangle dr. \end{aligned} \quad (2.2)$$

It is implicit in the definition that the process  $\langle \xi_t, \varphi \rangle$  has continuous trajectories.

Here is the main result about stochastic Euler vorticity equation.

**Theorem 2.9.** *Given  $\xi_0$  in  $L^\infty(\mathbb{T}^2)$  and the cylindrical Brownian motion  $W$  (with the associated filtration), the stochastic Euler vorticity equation (2.1) admits a unique  $L^\infty$  distributional solution.*

**Remark 2.10.** *Notice that the filtration is given a-priori. Thus both the existence and the uniqueness are in the strong sense: there exists a solution  $\xi$  adapted to the (completed) Brownian filtration (the smallest possible filtration) and any solution, defined on a possibly larger filtered space, must coincide with  $\xi$ . The same kind of existence and uniqueness will hold for every equation we will meet.*

Theorem 2.9 will be proved by solving the associated non-local SDE:

$$\begin{aligned} \Phi_t(x) &= x + \int_0^t \int_{\mathbb{T}^2} K(\Phi_r(x) - \Phi_r(y)) \xi_0(y) dy dr \\ &\quad + \sum_k \int_0^t \sigma_k(\Phi_r(x)) dW_r^k. \end{aligned} \quad (2.3)$$

Notice that here the drift, namely

$$u^\Phi(t, x) = \int_{\mathbb{T}^2} K(x - \Phi_t(y)) \xi_0(y) dy, \quad (2.4)$$

depends on the whole flow.

**Definition 2.11.** *A (measurable) stochastic flow is a measurable map  $\Phi : [0, T] \times \mathbb{T}^2 \times \Omega \rightarrow \mathbb{T}^2$ . A stochastic continuous flow is a flow  $\Phi$  such that, for a.e.  $\omega$  in  $\Omega$ ,  $\Phi(\omega)$  is continuous in  $(t, x)$ . A stochastic continuous flow  $\Phi$  is said to be measure-preserving if, for a.e.  $\omega$ ,  $\Phi_t$  preserves the Lebesgue measure on  $\mathbb{T}^2$  for every  $t$ .*

**Theorem 2.12.** *Given  $\xi_0$  in  $L^\infty(\mathbb{T}^2)$  and the cylindrical Brownian motion  $W$  (with the associated filtration), there exists a unique measure-preserving stochastic flow which solves equation (2.3). This solution is a continuous flow  $\Phi$  of class  $C^\alpha$  in space and  $C^\beta$  in time, for some  $\alpha > 0$  and for every  $\beta < 1/2$ .*

## 2.1 The strategy

There are two ways to prove our results. We will develop mainly the one which requires the weakest regularity assumptions on the  $\sigma_k$ 's. This strategy will be as follows.

First we will prove that, for a log-Lipschitz random vector field  $u$ , the SDE

$$dX_t = u(X_t)dt + \sum_k \sigma_k(X_t)dW_t^k$$

admits a unique solution, given by a stochastic measure-preserving continuous flow (Lemma 4.5). This includes the case of a “linear” version of (2.3), where the drift is replaced by  $u^\psi$  for some fixed stochastic flow  $\psi$  (see also next paragraph for notation). Then, using an iteration scheme, we will build a unique solution to (2.3), reaching the assertion of Theorem 2.12.

In the subsequent section, we will use Theorem 2.12 to prove Theorem 2.9.

In the last section, we will show the second method: a “trick” allows us to reduce the stochastic case to a modified deterministic case. This seems to be more rapid but requires the  $\sigma_k$ 's to be at least  $C^2$  (at least if one wants to use classical results), while the first method requires only a Lipschitz hypothesis on the diffusion coefficients. That is why we will not develop this second method in all the details.

## 2.2 Log-Lipschitz property of $K$ and other useful facts

First we state the fundamental log-Lipschitz property for  $K$  and the drift  $u^\xi$ . The key inequality (2.6) is stated in [22, section 1.2], and it follows from standard estimates of the Green function  $G$  (see e.g. [3, section 4.2]); for completeness, we have recalled the proof in the appendix. For  $r \geq 0$ , call  $\gamma(r) = r(1 - \log r)1_{]0,1[}(r) + r1_{[1,+\infty[}(r)$ .

**Remark 2.13.** *The following elementary property of  $\gamma$  will be of use: for every  $0 < \varepsilon < 1/e$ , calling  $L_\varepsilon = -\log \varepsilon$ , we have*

$$\gamma(r) \leq L_\varepsilon r + \varepsilon, \quad \forall r \geq 0. \quad (2.5)$$

**Lemma 2.14.** *The map  $K$ , introduced before, is an  $L^p(\mathbb{T}^2)$  divergence-free (in the distributional sense) vector field, for every  $p < 2$ , and verifies*

$$\int_{\mathbb{T}^2} |K(x - y) - K(x' - y)| dy \leq C\gamma(|x - x'|), \quad \forall x, x' \in \mathbb{T}^2. \quad (2.6)$$

The divergence-free property is a consequence of the fact that  $K$  is orthogonal to a gradient of a scalar field.

**Corollary 2.15.** *For every  $w$  in  $L^\infty(\mathbb{T}^2)$ ,  $u^w = K * w$  is divergence-free and satisfies*

$$|u^w(x) - u^w(x')| \leq C\|w\|_{L^\infty}\gamma(|x - x'|), \quad \forall x, x' \in \mathbb{T}^2. \quad (2.7)$$

We will use also the following elementary result. We recall that, for a finite signed measure  $\mu$  on a space  $E$  and a measurable map  $F : E \rightarrow E'$ ,  $\nu = F_\# \mu$  denotes the image measure of  $\mu$  on  $E'$ , namely  $\nu(A) = \mu(F^{-1}(A))$  for every measurable set  $A$  in  $E'$ . Notice that  $\nu$  is a finite signed measure and that  $|\nu| \leq F_\# |\mu|$  (since  $|\nu|(A) \leq F_\# |\mu|(A)$  for every  $A$ ).

**Lemma 2.16.** *Let  $F$  be a measure preserving map on  $\mathbb{T}^2$  and let  $w$  be in  $L^\infty(\mathbb{T}^2)$ . Let  $\mu$  the (signed) measure on  $\mathbb{T}^2$  with density  $w$  (with respect to the Lebesgue measure) and define  $\nu = F_\# w$ . Then  $\nu$  has a density (denoted by  $v$ ) with respect to Lebesgue measure and  $\|v\|_{L^\infty} \leq \|w\|_{L^\infty}$ .*

*Proof.* It is enough to prove the Lemma when  $w$  is nonnegative. Since  $F$  is measure-preserving, if  $A$  is a set of zero Lebesgue measure, then  $\mathcal{L}^2\{F \in A\} = \mathcal{L}^2(A) = 0$ , and so  $\int_A d\nu = \int_{\mathbb{T}^2} 1_A(F)w dx = 0$ . So  $\nu$  admits a (nonnegative) density  $v$ . Now, taking  $\varepsilon > 0$ ,  $B = \{v > \|w\|_{L^\infty} + \varepsilon\}$ , we have

$$\begin{aligned} (\|w\|_{L^\infty} + \varepsilon)\mathcal{L}^2(B) &\leq \int_{\mathbb{T}^2} 1_B v dx = \int_{\mathbb{T}^2} 1_B(F)w dx \leq \\ &\leq \|w\|_{L^\infty}\mathcal{L}^2\{F \in B\} = \|w\|_{L^\infty}\mathcal{L}^2(B), \end{aligned}$$

which implies that  $\mathcal{L}^2(B) = 0$ . By arbitrariness of  $\varepsilon$ , we get  $\|v\|_{L^\infty} \leq \|w\|_{L^\infty}$ .  $\square$

Given Lemma 2.16, we will use often  $v = F_\# w$  instead of  $\nu = F_\# \mu$ . Finally some other notation. Let  $\psi$  a measurable measure-preserving flow on  $\mathbb{T}^2$ . With the notation in the previous section define  $\xi_t^\psi = (\psi_t)_\# \xi_0$  (which is in  $L^\infty([0, T] \times \mathbb{T}^2)$  by Lemma 2.16) and  $u^\psi = u^{\xi^\psi}$ , which also reads

$$u^\psi(t, x) = \int_{\mathbb{T}^2} K(x - \psi_t(y))\xi_0(y)dy.$$

As already noticed, the SDE (2.3) reads as

$$\Phi_t(x) = x + \int_0^t u_r^\Phi(\phi_r(x))dr + \sum_k \int_0^t \sigma_k(\Phi_r(x))dW_r^k.$$

### 3 The deterministic case

We first treat the deterministic case, in order to show the basic ideas. The scheme of the proof, strongly inspired by [22], is a suitable rewriting of [22], convenient for generalization to the stochastic case.

Euler flows in 2D (on the torus  $\mathbb{T}^2$ ) are described by the following non-local ODE:

$$\Phi_t(x) = x + \int_0^t \int_{\mathbb{T}^2} K(\Phi_s(x) - \Phi_s(y)) \xi_0(y) dy. \quad (3.1)$$

Equation (3.1) reads as  $\dot{\Phi} = u^\Phi(\Phi)$  (with initial condition  $\Phi_0 = id$ ), notice that the drift is log-Lipschitz. That is why we consider the auxiliary equation (linear problem):

$$X_t^x = x + \int_0^t u(s, X_s^x) ds, \quad (3.2)$$

where  $u$  is a fixed measurable vector field with the following property: for every  $t, x, y$ ,

$$|u(t, x) - u(t, y)| \leq C\gamma(|x - y|) \quad (3.3)$$

for some  $C$  independent of  $t, x, y$ .

**Lemma 3.1.** *For every initial datum  $x$ , equation (3.2) has a unique solution. This solution is described by a (unique) flow  $\psi$  of measure-preserving homeomorphisms of class  $C^\alpha$  in space and Lipschitz in time, with  $\alpha = \exp[-C\|\xi_0\|_{L^\infty}T]$ .*

*Proof.* The existence of a global solution (in  $\mathbb{R}^2$ ) to (3.2) follows from the Peano Theorem, since  $u$  is continuous bounded. The uniqueness holds by the Osgood criterion (since  $\int_0^\varepsilon \gamma(r)^{-1} dr = +\infty$ ) or even by the Hölder estimate below (simply take  $x = y$ ).

The Lipschitz continuity in time follows by boundedness of  $u$ . As for Hölder continuity, property (3.3) implies that, for every  $x$  and  $y$ ,

$$|\psi_t(x) - \psi_t(x')| \leq |x - x'| + C \int_0^t \gamma(|\psi_s(x) - \psi_s(x')|) ds.$$

By a comparison result,  $|\psi_t(x) - \psi_t(x')| \leq z(t, |x - x'|)$ , where

$$z(t, z_0) = z_0^{\exp[-Ct]} e^{1 - \exp[Ct]} 1_{t < t_0} + \exp[C(t - t_0)] 1_{t \geq t_0} \quad (3.4)$$



is the unique solution to  $z_t = z_0 + \int_0^t C\gamma(z_s)ds$  ( $t_0$  is the time such that  $z(t_0) = 1$ ). This gives the desired regularity. The invertibility and the continuity of the inverse map are due to the classical cocycle law, so that the inverse flow of  $\psi_t$  is  $\psi_{-t}$ . The measure-preserving property follows by a simple approximation argument, see the proof of Lemma 4.5 in the stochastic case.  $\square$

Now we use the Picard iteration scheme to prove the existence and the uniqueness of solutions to (3.1). Consider the set

$$M = \left\{ \psi : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2 \mid \begin{aligned} &\psi \text{ measurable, } \sup_{[0, T]} \int_{\mathbb{T}^2} |\psi_t(x)| dx < +\infty, \\ &\psi_t \text{ measure-preserving for a.e. } t \end{aligned} \right\}.$$

It is a complete metric space, endowed with the distance  $\text{dist}(\psi^1, \psi^2) = \sup_{[0, T]} \int_{\mathbb{T}^2} |\psi_t^1(x) - \psi_t^2(x)| dx$ . For any  $\psi$  in  $M$ , define  $G(\psi)$  as the unique flow solution to (3.2) with  $u = u^\psi$ , i.e.

$$\frac{d}{dt} G(\psi) = u^\psi(G(\psi)).$$

By the previous Lemma  $G$  takes values in  $M$ .

**Lemma 3.2.** *For every  $\varepsilon > 0$ , for every two flows  $\psi^1, \psi^2$  in  $M$ , the following estimates hold:*

$$\begin{aligned} &\int_{\mathbb{T}^2} |G(\psi^1)_t(x) - G(\psi^2)_t(x)| dx \\ &\leq C \|\xi_0\|_{L^\infty} \int_0^t \gamma \left( \int_{\mathbb{T}^2} |\psi_s^1(x) - \psi_s^2(x)| dx \right) ds \\ &+ C \|\xi_0\|_{L^\infty} \int_0^t \gamma \left( \int_{\mathbb{T}^2} |G(\psi^1)_s(x) - G(\psi^2)_s(x)| dx \right) ds, \end{aligned} \quad (3.5)$$

$$\begin{aligned} &\int_{\mathbb{T}^2} |G(\psi^1)_t - G(\psi^2)_t| dx \\ &\leq C \|\xi_0\|_{L^\infty} L_\varepsilon \int_0^t \int_{\mathbb{T}^2} |G(\psi^1)_s - G(\psi^2)_s| dx ds \\ &+ C \|\xi_0\|_{L^\infty} L_\varepsilon \int_0^t \int_{\mathbb{T}^2} |\psi_s^1 - \psi_s^2| dx ds + 2C \|\xi_0\|_{L^\infty} t\varepsilon. \end{aligned} \quad (3.6)$$

*Proof.* We have

$$\int_{\mathbb{T}^2} |G(\psi^1)_t(x) - G(\psi^2)_t(x)| dx \leq \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |K(G(\psi^1)_s(x) - \psi_s^1(y)) - K(G(\psi^2)_s(x) - \psi_s^2(y))| dx dy ds.$$

In order to use (2.6), we add and subtract  $K(G(\psi^1)_s(x) - \psi_s^2(y))$  to the integrand of the right-hand side. Thus we get

$$\begin{aligned} \int_{\mathbb{T}^2} |G(\psi^1)_t(x) - G(\psi^2)_t(x)| dx &\leq \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left[ |K(G(\psi^1)_s(x) - \psi_s^1(y)) - K(G(\psi^1)_s(x) - \psi_s^2(y))| \right. \\ &\quad \left. + |K(G(\psi^1)_s(x) - \psi_s^2(y)) - K(G(\psi^2)_s(x) - \psi_s^2(y))| \right] dx dy ds \\ &\leq \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left[ |K(x - \psi_s^1(y)) - K(x - \psi_s^2(y))| \right. \\ &\quad \left. + |K(G(\psi^1)_s(x) - y) - K(G(\psi^2)_s(x) - y)| \right] dx dy ds \\ &\leq C \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \gamma(|\psi_s^1(y) - \psi_s^2(y)|) dy ds \\ &\quad + C \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \gamma(|G(\psi^1)_s(x) - G(\psi^2)_s(x)|) dx ds, \end{aligned}$$

where in the second passage we used the measure-preserving property. Finally, by the Jensen inequality applied to the concave function  $\gamma$ , we have

$$\begin{aligned} \int_{\mathbb{T}^2} |G(\psi^1)_t(x) - G(\psi^2)_t(x)| dx &\leq C \|\xi_0\|_{L^\infty} \int_0^t \gamma \left( \int_{\mathbb{T}^2} |\psi_s^1(x) - \psi_s^2(x)| dx \right) ds \\ &\quad + C \|\xi_0\|_{L^\infty} \int_0^t \gamma \left( \int_{\mathbb{T}^2} |G(\psi^1)_s(x) - G(\psi^2)_s(x)| dx \right) ds, \end{aligned}$$

that is the first estimate (3.5). Now we apply property (2.5):

$$\begin{aligned}
& \int_{\mathbb{T}^2} |G(\psi^1)_t(x) - G(\psi^2)_t(x)| dx \\
& \leq C \|\xi_0\|_{L^\infty} L_\varepsilon \int_0^t \int_{\mathbb{T}^2} |\psi_s^1(x) - \psi_s^2(x)| dx ds + \\
& \quad + C \|\xi_0\|_{L^\infty} L_\varepsilon \int_0^t \int_{\mathbb{T}^2} |G(\psi^1)_s(x) - G(\psi^2)_s(x)| dx ds + 2C \|\xi_0\|_{L^\infty} t \varepsilon,
\end{aligned}$$

i.e. the second estimate (3.6).  $\square$

The following continuity result is a consequence of the previous Lemma.

**Corollary 3.3.** *The map  $G : M_T \rightarrow M_T$  is continuous. In fact, it is locally Hölder continuous.*

*Proof.* Call  $w_t = \int_{\mathbb{T}^2} |G(\psi^1)_t(x) - G(\psi^2)_t(x)| dx$ . Then Lemma 3.2, together with monotonicity of  $\gamma$ , gives

$$\begin{aligned}
w_t & \leq C \|\xi_0\|_{L^\infty} T \gamma \left( \sup_{s \in [0, T]} \int_{\mathbb{T}^2} |\psi_s^1(x) - \psi_s^2(x)| dx \right) \\
& \quad + C \|\xi_0\|_{L^\infty} \int_0^t \gamma(w_s) ds.
\end{aligned}$$

Again a comparison theorem (recall the definition of  $z$  in (3.4), possibly with a different constant  $C$ ), we get the following estimate, valid when  $\text{dist}(\psi^1, \psi^2) < 1$  and  $t \in [0, T]$ :

$$\begin{aligned}
& \int_{\mathbb{T}^2} |G(\psi^1)_t(x) - G(\psi^2)_t(x)| dx \\
& \leq z \left( t, C \|\xi_0\|_{L^\infty} \gamma \left( \sup_{s \in [0, T]} \int_{\mathbb{T}^2} |\psi_s^1(x) - \psi_s^2(x)| dx \right) \right) \\
& \leq C' \left( \gamma \left( \sup_{s \in [0, T]} \int_{\mathbb{T}^2} |\psi_s^1(x) - \psi_s^2(x)| dx \right) \right)^{\exp[-Ct]}.
\end{aligned}$$

From this and the continuity of  $\gamma$ , we see that  $G$  is continuous on  $M_T$ . The Hölder continuity of  $G$  follows from the fact that  $\gamma$  is Hölder continuous.  $\square$

Now we define the approximating sequence for the solution to problem (3.1). Choose  $\psi_t^0 = I$ . For any  $n$ , put  $\psi^{n+1} = G(\psi^n)$  and denote  $\rho_t^n = \sup_{k \geq n} \int_{\mathbb{T}^2} |\psi_t^{n+1}(x) - \psi_t^n(x)| dx$ . Then the previous lemma gives immediately that for all  $n \in \mathbb{N}$

$$\rho_t^n \leq 2CL_\varepsilon \int_0^t \rho_s^{n-1} ds + 2Ct\varepsilon. \quad (3.7)$$

By Lemma A.1 we infer that

$$\sup_{[0,T]} \rho_t^n \leq \frac{(2C\|\xi_0\|_{L^\infty}T)^n}{\sqrt{2\pi n}} \sup_{[0,T]} \rho_t^0 + 2C\|\xi_0\|_{L^\infty}T \exp[n(2C\|\xi_0\|_{L^\infty}T - 1)] \quad (3.8)$$

and so, provided  $\alpha := 2C\|\xi_0\|_{L^\infty}T < 1$ , there exists a unique  $\psi \in M_T$  such that the sequence  $(\psi^n)_n$  converges in  $M_T$  to  $\psi$ . By Corollary 3.3 it follows that  $G(\psi) = \psi$ .

We are ready to prove:

**Theorem 3.4.** *There exists a unique solution in  $M$  to equation (3.1), which is a flow  $\Phi$  of measure-preserving homeomorphisms of class  $C^\alpha$  in space and Lipschitz in time.*

*Proof.* For  $T$  small enough, the existence has been proven. The uniqueness follows by applying the previous iterative procedure to two solutions  $\Phi^1, \Phi^2$ . Indeed from (3.8), we get

$$\text{dist}(\Phi^1, \Phi^2) = \text{dist}(G(\Phi^1), G(\Phi^2)) \leq \alpha^n \text{dist}(\Phi^1, \Phi^2) + \alpha e^{-n(1-\alpha)},$$

for any integer  $n$ . If  $\alpha < 1$  (i.e.  $T$  is small) and  $n$  is large, we get  $\text{dist}(\Phi^1, \Phi^2) = 0$ .

The global existence follows by iteration on time, but it is not the most classical argument, since the equation does not remain the same starting from a time  $t_0 > 0$ . More precisely, if we repeat the previous procedure, but taking  $\psi_t^1 = \Phi_t 1_{[0,T]} + \Phi_T 1_{]T,+\infty[}$ , we get that now, for  $t \leq T$ ,  $\psi_t^n = \Phi_t$  for every  $n$ . So, for  $t \leq T$ ,  $\rho_t^n = 0$  for every  $n$ , and, for  $t > T$ ,

$$\rho_t^n \leq 2CL_\varepsilon \int_T^t \rho_s^{n-1} ds + 2C(t-T)\varepsilon,$$

where  $C$  is the same constant as in (3.7). So we can prove that  $\Phi$  can be extended, as solution to (3.1), until time  $2T$ . Iterating this procedure, we obtain the global existence.

Finally, the regularity properties hold since  $\Phi = G(\Phi)$  is in the image of  $G$ .  $\square$

**Remark 3.5.** *In case  $\xi_0$  is more smooth, more regularity of  $\Phi$  can be obtained, using the usual iterative scheme: if  $\Phi$  has some regularity, then  $u^\Phi$  has more regularity, which implies again an improvement of regularity of  $\Phi$ , and so on.*

## 4 The stochastic case

Now we prove the existence and the uniqueness of a stochastic continuous flow solving equation (2.3). Notice that, differently from the classical (linear) case, the drift depends on the whole flow, so Kunita's theory is not (at least easily) applicable.

We try to mimic the previous reasoning in the deterministic case. The last part, the iterative procedure from the proof of Theorem 3.4, works in this simple way. First we get a generalized Lemma 3.1 (with Itô formula to treat the modulus of the difference of two flows), then we repeat the scheme and obtain a measurable flow solution to (2.3).

The main difficulty is in the first part, precisely in the generalization of Lemma 3.1 to stochastic continuous flows (remember that we need a continuity property for  $\omega$  fixed). In order to get rid of the first difficulty, we will apply Kolmogorov test, in the spirit of Kunita's results. For this we need some estimates on the linear equation.

### 4.1 The linear stochastic equation

Consider the following SDE ("linear" problem):

$$dX_t = u_t(X_t)dt + \sum_k \sigma_k(X_t)dW_t^k, \quad (4.1)$$

where  $u$  is a random vector field with the following properties: for every  $x$ ,  $(t, \omega) \rightarrow (t, x, \omega)$  is a progressively measurable process and, for every  $t, x, y, \omega$ ,

$$|u(t, x, \omega) - u(t, y, \omega)| \leq C\gamma(|x - y|) \quad (4.2)$$

for some  $C$  independent of  $t, x, y, \omega$ . These properties imply that, if  $X$  is a progressively measurable process with values in  $\mathbb{T}^2$ , then  $u(t, X_t)$  is progressively measurable too.

In the following, every constant will be denoted by  $C$ .

**Lemma 4.1.** *Let  $X, Y$  be two solutions of (4.1) starting from  $x, x'$  resp.. Then, for any  $p \geq 2$ , it holds*

$$E[|X_t - Y_{t'}|^p] \leq C(|x - x'|^{p \exp[-CT]} + |t - t'|^{p/2}). \quad (4.3)$$

*Proof.* It is enough to prove the formula in the two particular cases  $t = t'$  and  $x = x'$ . Fix  $t = t'$ . By the Itô formula (applied to  $f(x) = |x|^p$ ), calling  $Z = X - Y$ , we have

$$\begin{aligned} d[|Z|^p] &= p|Z|^{p-2}Z \cdot (u(X) - u(Y)) dt \\ &+ \left[ \sum_k p|Z|^{p-2}|\sigma_k(X) - \sigma_k(Y)|^2 \right] dt \\ &+ \left[ \sum_k p(p-2)|Z|^{p-4}|Z \cdot (\sigma_k(X) - \sigma_k(Y))|^2 \right] dt + \\ &+ \sum_k p|Z|^{p-2}Z \cdot (\sigma_k(X) - \sigma_k(Y))dW^k. \end{aligned}$$

We take the expectation and use Lipschitz continuity of  $\sigma_k$ 's and log-Lipschitz property of  $u$ :  $p|Z|^{p-1}|u(X) - u(Y)| \leq pC|Z|^p(1 - 1_{|Z|<1} \log |Z|) \leq pC\gamma(|Z|^p)$ . Then

$$E[|Z_t|^p] \leq |x - x'|^p + C \int_0^t E[\gamma(|Z_s|^p)] ds + C \int_0^t E[|Z_s|^p] ds,$$

from which, using Jensen inequality for the concave function  $\gamma$ , we obtain

$$E[|Z_t|^p] \leq |x - x'|^p + C \int_0^t \gamma(E[|Z_s|^p]) ds + C \int_0^t E[|Z_s|^p] ds.$$

By a comparison principle,  $E[|Z_t|^p] \leq v(t, |x - x'|^p)$ , where

$$v(t, v_0) = (v_0)^{\exp[-Ct]} e^{1 - \exp[-Ct]} 1_{v_0 < 1} + e^{2C(t-t_0)} 1_{v_0 \geq 1} \quad (4.4)$$

is the unique solution to  $v_t = v_0 + C \int_0^t (\gamma(v_s) + v_s) ds$ . We are done for the first case.

Now put  $\eta = \eta', t' < t$ . By the boundedness of  $u$  and  $\sigma_k$ 's, using the Hölder and the Burkholder inequalities, we get

$$\begin{aligned} E[|X_t - X_{t'}|^p] &\leq 2^{p-1} E \left[ \left| \int_{t'}^t u(X_r) dr \right|^p + \left| \sum_k \int_{t'}^t \sigma_k(X_r) dW_r^k \right|^p \right] \\ &\leq C(|t - t'|^p + |t - t'|^{p/2}). \end{aligned}$$

The proof is complete.  $\square$

This will be enough to get the uniqueness and the continuity, but we still need the existence. For this, we will use a generalization of the previous lemma, exhibiting a Cauchy sequence of solutions of approximating equations. Let  $\rho$  be a  $C_c^\infty(\mathbb{R}^2)$  function, define  $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(\varepsilon^{-1}x)$ ; consider the standard mollification of  $u$ :  $u^\varepsilon(t, x, \omega) = u(t, \cdot, \omega) * \rho_\varepsilon(x)$ , for  $x \in \mathbb{T}^2$  (the convolution must be understood on the whole  $\mathbb{R}^d$ , where  $u$  is extended by periodicity). Notice that, since by (4.2) the field  $u$  is continuous and bounded in  $x$ , uniformly with respect to  $t$  and  $\omega$ , we get that  $(u^\varepsilon)_\varepsilon$  converges to  $u$  uniformly in  $(t, x, \omega)$ : that is, we can find a continuous function  $\theta : [0, 1] \rightarrow [0, +\infty[$ , with  $\theta(0) = 0$ , such that, for every  $\varepsilon > 0$ ,  $\delta > 0$ ,

$$\sup_{[0, T] \times \mathbb{T}^2 \times \Omega} |u^\varepsilon - u^\delta| \leq \theta(|\varepsilon - \delta|). \quad (4.5)$$

Moreover, Corollary 3.3 holds uniformly in  $\varepsilon$ :

$$\sup_{\varepsilon > 0} |u^\varepsilon(t, x) - u^\varepsilon(t, x')| \leq C\gamma(|x - x'|). \quad (4.6)$$

Similarly, we define  $\sigma_k^\varepsilon(t, x) := \sigma_k(t, \cdot) * \rho_\varepsilon(x)$ ; since the  $\sigma_k$ 's are Lipschitz-continuous (more precisely, by Condition 2.2), we get (possibly for another  $\theta$ , with the same properties as above)

$$\sup_{[0, T] \times \mathbb{T}^2} \sum_k |\sigma_k^\varepsilon - \sigma_k^\delta|^2 \leq \theta(|\varepsilon - \delta|), \quad (4.7)$$

$$\sup_{\varepsilon > 0} \sum_k |\sigma_k^\varepsilon(t, x) - \sigma_k^\varepsilon(t, x')|^2 \leq C|x - x'|^2. \quad (4.8)$$

**Lemma 4.2.** *For any  $\varepsilon > 0$ , let  $\psi^\varepsilon$  be the stochastic continuous flow solution to*

$$dX_t^\varepsilon = u_t^\varepsilon(X_t^\varepsilon)dt + \sum_k \sigma_k^\varepsilon(X_t^\varepsilon)dW_t^k. \quad (4.9)$$

*Then, for any  $p \geq 2$ , for every  $\varepsilon, \delta$ , for every  $x, x'$  in  $\mathbb{T}^2$ , it holds*

$$\sup_{[0, T]} E[|\psi_t^\varepsilon(x) - \psi_t^\delta(x')|^p] \leq C(|x - x'|^p + C\theta(\varepsilon - \delta))^{\exp[-Ct]}$$

*for some  $C > 0$  (independent of  $\varepsilon, \delta, x, x'$ ). In particular,  $(\psi^\varepsilon)_\varepsilon$  is a Cauchy sequence in  $C([0, T] \times \mathbb{T}^2; L^p(\Omega))$ .*

**Remark 4.3.** For every  $\varepsilon > 0$ , for every initial datum, equation (4.9) has a unique solution, which can be represented by a stochastic continuous flow  $\psi^\varepsilon$  of  $C^1$  maps. Indeed, by the boundedness of  $u$ , the  $C^1$  norm of  $u^\varepsilon$  is uniformly bounded, and Kunita's theory applies. Notice that here we need Kunita's result with a stochastic drift, namely [20], Theorem 4.6.5.

**Remark 4.4.** Again for  $\varepsilon > 0$ , since the stochastic integral is of Stratonovich type (which we have written in Itô form), usual calculus rules give the standard equation for the Jacobian, which depends only on the divergence of the vector fields. Since  $u^\varepsilon$  and  $\sigma_k^\varepsilon$ 's are divergence free, the Jacobian turns out to be constant and so the stochastic flow is measure-preserving.

*Proof.* First we notice that, for  $p \geq 2$ ,  $E[|\psi_t^\varepsilon(x)|^p]$  is bounded by a constant independent of  $\varepsilon$ ,  $t$  and  $x$  (simply estimate  $|u^\varepsilon(X^\varepsilon)|$  and  $|\sigma_k(X^\varepsilon)|$  with the sup-norms of  $u$  and  $\sigma_k$  and use Hölder and Burkholder inequalities). Similarly, one sees that  $\psi^\varepsilon$  is in  $C([0, T] \times \mathbb{T}^2; L^p(\Omega))$  for every  $\varepsilon > 0$ . By Itô formula (applied to  $f(x) = |x|^p$ ), calling  $Z = \psi_t^\varepsilon(y) - \psi_t^\delta(x)$ , we have

$$\begin{aligned} d[|Z|^p] &= p|Z|^{p-2}Z \cdot (u^\varepsilon(\psi^\varepsilon(x)) - u^\delta(\psi^\delta(x'))))dt \\ &+ \left[ \sum_k p|Z|^{p-2}|\sigma_k^\varepsilon(\psi^\varepsilon(x)) - \sigma_k^\delta(\psi^\delta(x'))|^2 \right. \\ &+ \left. \sum_k p(p-2)|Z|^{p-4}|Z \cdot (\sigma_k^\varepsilon(\psi^\varepsilon(x)) - \sigma_k^\delta(\psi^\delta(x')))|^2 \right]dt \\ &+ \sum_k p|Z|^{p-2}Z \cdot (\sigma_k^\varepsilon(\psi^\varepsilon(x)) - \sigma_k^\delta(\psi^\delta(x'))))dW^k. \end{aligned}$$

The difficult term is  $u^\varepsilon(\psi^\varepsilon(x)) - u^\delta(\psi^\delta(x'))$ . For this, by (4.5) and (4.6), we have

$$\begin{aligned} &|u^\varepsilon(\psi^\varepsilon(x)) - u^\delta(\psi^\delta(x'))| \\ &\leq |u^\varepsilon(\psi^\varepsilon(x)) - u^\delta(\psi^\varepsilon(x))| + |u^\delta(\psi^\varepsilon(x)) - u^\delta(\psi^\delta(x'))| \\ &\leq \theta(\varepsilon - \delta) + C\gamma(|Z|). \end{aligned}$$

The terms with  $\sigma_k^\varepsilon$  are easier: by (4.7) and (4.8), we have

$$\begin{aligned} &\sum_k |\sigma_k^\varepsilon(\psi^\varepsilon(x)) - \sigma_k^\delta(\psi^\delta(x'))|^2 \\ &\leq 2 \sum_k [|\sigma_k^\varepsilon(\psi^\varepsilon(x)) - \sigma_k^\delta(\psi^\varepsilon(x))|^2 + |\sigma_k^\delta(\psi^\varepsilon(x)) - \sigma_k^\delta(\psi^\delta(x'))|^2] \\ &\leq \theta(\varepsilon - \delta) + C|Z|^2. \end{aligned}$$



So, proceeding as before, using concavity of  $\gamma$  and uniform boundedness of  $E[|Z|^{p-1}]$  and  $E[|Z|^{p-2}]$ , we get

$$E[|Z|_t^p] \leq |x - x'|^p + C\theta(|\varepsilon - \delta|) + \int_0^t \gamma(E[|Z|_s^p])ds + C \int_0^t E[|Z|_s^p]ds.$$

We conclude that

$$\sup_{[0,T]} E[|Z|_t^p] \leq C \left( |x - x'|^p + C\theta(|\varepsilon - \delta|) \right)^{\exp[-Ct]},$$

which implies that, if  $x = x'$ , the sequence  $(\psi^\varepsilon)_\varepsilon$  is Cauchy in the space  $C([0, T] \times \mathbb{T}^2; L^p(\Omega))$ .  $\square$

**Lemma 4.5.** *Equation (4.1) has a unique solution, for every deterministic initial datum. This solution is described by a (unique) stochastic measure-preserving continuous flow  $\psi$  of class  $C^\alpha$  in space, for some  $\alpha > 0$ , and  $C^\beta$  in time, for every  $\beta < 1/2$ .*

*Proof.* By the previous Lemma, for every  $x$ , there exists the limit (in  $C([0, T]; L^p(\Omega))$ )  $X$  of the approximating processes  $X^\varepsilon = \psi^\varepsilon(x)$ 's. Then equation (4.9) passes to the limit, because the coefficients are continuous bounded, and we get that  $X$  solves (4.1). The uniqueness follows from Lemma 4.1, with  $x = y$ . The Hölder continuity property is a consequence of the Kolmogorov criterion, applied again to (4.3). Indeed we get that  $\psi$  is  $\alpha$ -Hölder continuous in space, for every  $\alpha < e^{-CT} - 2/p$ , and  $\beta$ -Hölder continuous in time, for every  $\beta < 1/2 - 1/p$ , so for every  $\beta < 1/2$ .

As for measure-preserving property, we will prove that, for every bounded measurable  $F : \Omega \rightarrow \mathbb{R}$ , every bounded measurable  $h : [0, T] \rightarrow \mathbb{R}$  and every continuous bounded  $g : \mathbb{T}^2 \rightarrow \mathbb{R}$ ,

$$\int_0^T h(t) E \left[ F \int_{\mathbb{T}^2} g(\psi_t(x)) dx \right] dt = \int_0^T h(t) E \left[ F \int_{\mathbb{T}^2} g(x) dx \right] dt. \quad (4.10)$$

This will prove that, for a.e.  $(t, \omega)$ ,  $\psi_t(\omega)$  is measure-preserving; by continuity in  $(t, x)$  at  $\omega$  fixed, this implies easily that, for a.e.  $\omega$ ,  $\psi_t(\omega)$  is measure-preserving for every  $t$ . Since the approximating flows  $\psi^\varepsilon$ 's are measure-preserving (remember Remark 4.4), equality (4.10) holds for the  $\psi^\varepsilon$ 's. By convergence in  $L^p$ , we can find a subsequence  $\psi^{\varepsilon_n}$  such that  $(\psi^{\varepsilon_n})_n$  converges to  $\psi$  for a.e.  $(t, x, \omega)$ . Passing to the limit along this subsequence (using dominated convergence theorem), we get (4.10) for  $\psi$ . The proof is complete.  $\square$

**Remark 4.6.** *With a small effort, one could also show the injectivity of  $\psi_t(\omega)$  for all  $t$ , for a.e.  $\omega$  (essentially, one has to extend Lemma 4.1 to negative  $p$  and use Kolmogorov criterion for  $|\psi_t(x) - \psi_t(y)|^{-1}$ ). Surjectivity and continuity of the inverse map follow from the continuity and the measure-preserving property. The range of a measure-preserving continuous map is a compact set, whose complement (an open set) is Lebesgue-negligible. Thus this range must be the whole  $\mathbb{T}^2$ . Thus the flow is actually a flow of homeomorphisms.*

**Corollary 4.7.** *Let  $\xi$  be an element of  $L^\infty([0, T] \times \mathbb{T}^2 \times \Omega)$ . Then equation (4.1) with  $u = u^\xi$  has a unique solution, for every deterministic initial datum, which enjoys the properties in Lemma 4.5.*

## 4.2 Stochastic Euler flows

The rest of the section goes on in analogy with the deterministic case. We define a space

$$SM = \left\{ \psi : [0, T] \times \mathbb{T}^2 \times \Omega \rightarrow \mathbb{T}^2 : \psi \text{ measurable , } \right. \\ \left. \sup_{[0, T]} \int_{\mathbb{T}^2} E[|\psi_t(x)|] dx < +\infty, \psi_t \text{ meas.-pres. for a.e. } (t, \omega) \right\}.$$

For a given measure-preserving stochastic flow  $\psi$  in  $SM$ , we call  $G(\psi)$  the unique solution to the SDE (4.1) with  $u = u^\psi$ , i.e.

$$u^\psi(t, x) = \int_{\mathbb{T}^2} K(\psi_r(x) - \psi_r(y)) \xi_0(y) dy.$$

**Lemma 4.8.** *For every  $\varepsilon > 0$  (small enough), for every  $\psi^1, \psi^2$  flows in  $SM$ ,*

the following estimates hold (possibly enlarging  $C$ ):

$$\begin{aligned}
& \int_{\mathbb{T}^2} E|G(\psi^1)_t(x) - G(\psi^2)_t(x)|dx \\
& \leq C \int_0^t \gamma \left( \int_{\mathbb{T}^2} E|\psi_s^1(x) - \psi_s^2(x)|dx \right) ds \\
& + C \int_0^t \gamma \left( \int_{\mathbb{T}^2} E|G(\psi^1)_s(x) - G(\psi^2)_s(x)|dx \right) ds + \\
& + C \int_0^t \int_{\mathbb{T}^2} E|G(\psi^1)_s(x) - G(\psi^2)_s(x)|dx ds, \\
& \int_{\mathbb{T}^2} E|G(\psi^1)_t - G(\psi^2)_t|dx \leq CL_\varepsilon \int_0^t \int_{\mathbb{T}^2} E|G(\psi^1)_s - G(\psi^2)_s|dx ds \\
& + CL_\varepsilon \int_0^t \int_{\mathbb{T}^2} E|\psi_s^1 - \psi_s^2|dx ds + Ct\varepsilon.
\end{aligned}$$

*Proof.* We would like to apply Itô formula to the modulus function and get an estimate for  $|G(\psi^1)_t(x) - G(\psi^2)_t(x)|$ . Since the modulus is not  $C^2$ , we use the approximate functions  $f_\delta(x) = (|x|^2 + \delta)^{1/2}$ , for  $\delta > 0$ . Calling  $Z = G(\psi^1)_t(x) - G(\psi^2)_t(x)$ , we have

$$\begin{aligned}
d[f_\delta(Z)] &= f_\delta(Z)^{-1} Z \cdot [u^{\psi^1}(G(\psi^1)) - u^{\psi^2}(G(\psi^2))]dt \\
&+ \sum_k f_\delta(Z)^{-1} |\sigma_k(G(\psi^1)) - \sigma_k(G(\psi^2))|^2 dt \\
&+ \sum_k f_\delta(Z)^{-3} [(G(\psi^1) - G(\psi^2)) \cdot (\sigma_k(G(\psi^1)) - \sigma_k(G(\psi^2)))]^2 dt \\
&+ \sum_k f_\delta(Z)^{-1} Z \cdot [\sigma_k(G(\psi^1)) - \sigma_k(G(\psi^2))]dW.
\end{aligned}$$

Taking the expectation and using the Lipschitz property of  $\sigma$ , since  $f_\delta(x) \geq |x|$ , we get

$$\begin{aligned}
E[|Z_t|] &\leq \int_0^t E[|u_s^{\psi^1}(G(\psi^1)_s(x)) - u_s^{\psi^2}(G(\psi^2)_s(x))|]ds \\
&+ C \int_0^t E[|Z_s|]ds.
\end{aligned}$$

The rest of the proof follows the lines of Lemma 3.1: we estimate  $\int_{\mathbb{T}^2} |u_s^{\psi^1}(G(\psi^1)_s(x)) - u_s^{\psi^2}(G(\psi^2)_s(x))|dx$  and use Jensen inequality to pass  $\gamma$  outside the integral in

$x$  and outside the expectation. The second inequality is a consequence of the first (possibly enlarging  $C$ , the last term in the first estimate is incorporated automatically).  $\square$

The iteration scheme is completely similar to the one in the deterministic case: we consider  $\psi_t^0(x) = x$ ,  $\psi^{n+1} = G(\psi^n)$ ,  $\rho_t^n = \sup_{k \geq n} \int_{\mathbb{T}^2} E|\psi_t^{n+1}(x) - \psi_t^n(x)|dx$  and proceed as before, getting a limit flow  $\Phi$  in  $SM$ , for  $T$  small enough. Such a flow solves (2.3), because  $G$  is continuous in  $SM$ : indeed, from Lemma 4.8 again by comparison,

$$\begin{aligned} \int_{\mathbb{T}^2} E|G(\psi^1)_t(x) - G(\psi^2)_t(x)|dx \\ \leq v\left(t, CT\gamma\left(\sup_{s \in [0, T]} \int_{\mathbb{T}^2} E|\psi_s^1(x) - \psi_s^2(x)|dx\right)\right) \\ \leq C'\left(\gamma\left(\sup_{s \in [0, T]} \int_{\mathbb{T}^2} E|\psi_s^1(x) - \psi_s^2(x)|dx\right)\right)^{\exp[-Ct]}. \end{aligned}$$

*Proof of Theorem 2.12.* Similar to the proof of Theorem 3.4. First, by iteration in time (as in the deterministic case), one gets a unique flow  $\Phi$  in  $SM$  solving (2.3), then this flow has the desired regularity property because it is in the image of  $G$ .  $\square$

## 5 The stochastic Euler vorticity equation

In this section we will prove Theorem 2.9. First we need the existence of solutions to the stochastic Euler vorticity equation (2.1).

**Proposition 5.1.** *Let  $\Phi$  be a solution to (2.3). For  $t \geq 0$ , define  $\xi_t = (\Phi_t)_\# \xi_0$ . Then  $\xi$  has a density (still denoted by  $\xi$ ) in  $L^\infty([0, T] \times \mathbb{T}^2 \times \Omega)$ , which is a distributional  $L^\infty$  solution to the stochastic Euler equation (2.1).*

*Proof.* Fix  $t > 0$  and the probabilistic datum  $\omega$  (omitted in the sequel). By Lemma 2.16, since  $\Phi_t$  is measure preserving,  $\xi_t$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{T}^2$  and  $\|\xi_t\|_{L^\infty} \leq \|\xi_0\|_{L^\infty}$ .

Let  $\varphi$  be a test function, Itô formula applied to  $\varphi(\Phi_t)$  gives

$$\begin{aligned} d[\varphi(\Phi_t)] &= u_t^\Phi(\Phi_t) \cdot \nabla \varphi(\Phi_t) dt + \sum_k \sigma_k(\Phi_t) \cdot \nabla \varphi(\Phi_t) dW_r^k \\ &+ \frac{1}{2} \text{tr}[a(\Phi_t) D^2 \varphi(\Phi_t)] dt. \end{aligned}$$

Now notice that, by definition of  $\xi_t$ ,  $u_t^\Phi = K * \xi_t$ ; so, integrating in  $\xi_0 dx$ , we get (2.2).  $\square$

For the proof of uniqueness, we will adapt a classical argument for transport equation. We first recall the idea in the case  $\sigma_k \equiv 0$  for simplicity. A formal application of the chain rule gives

$$\frac{d}{dt}\xi_t(\Phi_t) = \partial_t \xi_t(\Phi_t) + D\xi_t(\Phi_t) \frac{d\Phi_t}{dt} = (\partial_t \xi_t + u_t \cdot \nabla \xi_t)(\Phi_t) = 0.$$

This implies that  $\xi_t(\Phi_t) = \xi_0$ , so that  $\xi_t = \xi_0(\Phi_t^{-1})$  is completely determined by the flow. But we have used the chain rule for an object  $(\xi_t)$  which is not regular in general (and in fact there are counterexamples for irregular drifts). Thus we need to regularize  $\xi$ . This regularization  $\xi^\varepsilon$  solves a transport-type equation with an additional term, a commutator, which we need to control to conclude the argument. We use for this the argument in [11], [1], [2], where the commutator is an essential tool for uniqueness of the transport equation. First we need approximate identities. For this, let  $\rho$  be a  $C^\infty(\mathbb{R}^2)$  nonnegative even function, with support in  $[-1/2, 1/2]^2$  and  $\int_{\mathbb{R}^2} \rho dx = 1$ . For  $\varepsilon > 0$ , define  $\rho_\varepsilon(x) = \varepsilon^{-2} \rho(x/\varepsilon)$ . If  $f$  is an integrable function on  $\mathbb{T}^2$ ,  $f$  can be extended periodically to a locally integrable function on the whole  $\mathbb{R}^2$ , so that the convolution  $\rho_\varepsilon * f$  makes sense and is still a  $C^\infty$  periodic function. For a vector field  $v$  and a function  $w$  on the torus, we define formally the commutator as

$$[v \cdot \nabla, \rho_\varepsilon *]w := v \cdot \nabla(\rho_\varepsilon * w) - \rho_\varepsilon * (v \cdot \nabla w). \quad (5.1)$$

Suppose that  $v$  and  $w$  are integrable and  $v$  is divergence free. Then the expression above defines a measurable function on  $\mathbb{T}^2$ . Indeed, the following equalities hold in distribution (the functions being thought as extended to the whole  $\mathbb{R}^2$ ):

$$\rho_\varepsilon * (v \cdot \nabla w) = \rho_\varepsilon * \operatorname{div}(vw) = - \int_{\mathbb{R}^2} \nabla \rho_\varepsilon(z) \cdot v(\cdot - z) w(\cdot - z) dz. \quad (5.2)$$

Besides, by (5.1) and (5.2), the commutator reads

$$[v \cdot \nabla, \rho_\varepsilon *]w(x) = \int_{\mathbb{R}^2} (v(x) - v(x - z)) \cdot \nabla \rho_\varepsilon(z) w(x - z) dz.$$

With the change of variable  $y = z/\varepsilon$ ,  $x' = x'_\varepsilon = x - \varepsilon y$  we get

$$[v \cdot \nabla, \rho_\varepsilon *]w(x) = \int_{\mathbb{R}^2} \frac{v(x' + \varepsilon y) - v(x')}{\varepsilon} \cdot \nabla \rho(y) w(x') dy.$$

If  $v$  is in  $W^{1,1}(\mathbb{T}^2)$ , then, for every  $y$  in  $\mathbb{R}^2$ , for a.e.  $x'$  in  $\mathbb{T}^2$ ,  $v(x' + \varepsilon y) - v(x') = \varepsilon \int_0^1 Dv(x' + \xi \varepsilon y) y d\xi$ . Indeed, this is true for  $v^\delta = \rho_\delta * v$  and, for fixed  $y$ ,  $v^\delta(x' + \varepsilon y) - v^\delta(x') - \varepsilon \int_0^1 Dv^\delta(x' + \xi \varepsilon y) y d\xi$ , as function of  $x'$ , converges to 0 a.e. as  $\delta \rightarrow 0$  (possibly passing to a subsequence). So, in this case, the commutator has the following expression:

$$[v \cdot \nabla, \rho_\varepsilon *]w(x') = \int_{\mathbb{R}^2} \int_0^1 Dv(x' + \xi \varepsilon y) y d\xi \cdot \nabla \rho(y) w(x') dy. \quad (5.3)$$

**Lemma 5.2** (Commutator lemma). *Let  $p$  be in  $[1, +\infty[$ , let  $v$  be in  $W^{1,p}(\mathbb{T}^2)$  with zero divergence, let  $w$  be in  $L^\infty(\mathbb{T}^2)$ . Then*

$$\lim_{\varepsilon \rightarrow 0} [v \cdot \nabla, \rho_\varepsilon *]w = 0 \quad \text{in } L^p(\mathbb{T}^2)$$

and we have the inequality

$$\|[v \cdot \nabla, \rho_\varepsilon *]w\|_{L^p(\mathbb{T}^2)} \leq C \|Dv\|_{L^p(\mathbb{T}^2)} \|w\|_{L^\infty(\mathbb{T}^2)}.$$

*Proof.* The inequality follows integrating in  $x$  the  $p$ -power of the expression on the LHS of (5.3). Precisely, since  $\rho$  is supported on  $[-1/2, 1/2]^2$ , we have by Hölder inequality (remember  $x' = x + \varepsilon y$ )

$$\begin{aligned} & \int_{\mathbb{T}^2} |[v \cdot \nabla, \rho_\varepsilon *]w|^p dx \\ & \leq \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \int_0^1 |Dv(x' + \xi \varepsilon y)|^p d\xi |w(x')|^p dx' |y|^p |\nabla \rho(y)|^p dy \\ & \leq \|Dv\|_{L^p(\mathbb{T}^2)}^p \|w\|_{L^\infty(\mathbb{T}^2)}^p \int_{\mathbb{R}^2} |y|^p |\nabla \rho(y)|^p dy \end{aligned}$$

(the integral in  $x'$  should be on  $\mathbb{T}^2 - \varepsilon y$ , but by periodicity we can integrate on  $\mathbb{T}^2$  as well).

For the limit, it is enough to show that

$$L^p(\mathbb{T}^2)\text{-}\lim_{\varepsilon \rightarrow 0} [v \cdot \nabla, \rho_\varepsilon *]w = w(\cdot) \left( \int_{\mathbb{R}^2} Dv(\cdot) y \cdot \nabla \rho(y) dy \right).$$

Indeed, by the symmetry property of  $\rho$ ,  $\int_{\mathbb{R}^2} y_i \partial_j \rho(y) dy = -C \delta_{ij}$  (where  $C$  is independent of  $i$ ) and so  $\int_{\mathbb{R}^2} Dv(x) y \cdot \nabla \rho(y) dy = -C \operatorname{div} w = 0$ . By (5.3) we have

$$\int_{\mathbb{T}^2} \left| [v \cdot \nabla, \rho_\varepsilon *] w(x) - w(x) \left( \int_{\mathbb{R}^2} Dv(x) y \cdot \nabla \rho(y) dy \right) \right|^p dx \leq \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \int_0^1 |w(x') Dv(x' + \xi \varepsilon y) - w(x' + \varepsilon y) Dv(x' + \varepsilon y)|^p d\xi dx' |y|^p |\nabla \rho(y)|^p dy,$$

hence it is enough to prove that

$$\int_{\mathbb{T}^2} \int_0^1 |w(x') Dv(x' + \xi \varepsilon y) - w(x' + \varepsilon y) Dv(x' + \varepsilon y)|^p d\xi dx' \rightarrow 0$$

uniformly in  $y$ . Using the continuity of translations in  $L^p$  for the function  $w Dv$ , we need only to show that

$$\int_{\mathbb{T}^2} \int_0^1 |w(x') Dv(x' + \xi \varepsilon y) - w(x') Dv(x')|^p d\xi dx' \rightarrow 0.$$

Since  $w$  is in  $L^\infty$ , this follows from  $\int_{\mathbb{T}^2} \int_0^1 |Dv(x' + \xi \varepsilon y) - Dv(x')|^p d\xi dx' \rightarrow 0$ , which is again a consequence of continuity of translation in  $L^p$  applied to  $Dv$ .  $\square$

**Proposition 5.3.** *Let  $\xi$  be a (distributional)  $L^\infty$  solution to stochastic Euler vorticity equation. Let  $\Phi$  be a measure-preserving stochastic flow, which solves (4.1) with  $u = u^\xi$  (it exists by Corollary 4.7). Then  $\xi_t = (\Phi_t)_\# \xi_0$ .*

*Proof.* We will prove that  $\xi_t(\Phi_t) = \xi_0$  Lebesgue-a.e.. Having this, then, for every measurable bounded function  $\varphi$  on  $\mathbb{T}^2$ ,  $\langle \xi_t, \varphi \rangle = \langle \xi_t(\Phi_t), \varphi(\Phi_t) \rangle = \langle \xi_0, \varphi(\Phi_t) \rangle$  (in the first equality we used the measure-preserving property) and so  $\xi_t = (\Phi_t)_\# \xi_0$ .

As mentioned before, we need to consider  $\xi_t^\varepsilon = \xi_t * \rho_\varepsilon$  instead of  $\xi_t$ . Notice that, for every  $x$ ,  $\xi_t^\varepsilon(x) = \langle \xi_t, \rho_\varepsilon(x - \cdot) \rangle$ . So  $\xi^\varepsilon(x)$  is a progressively measurable process, with continuous trajectories, and stochastic Euler vorticity equation, applied to the test function  $\rho_\varepsilon(x - \cdot)$ , gives the following equality:

$$d\xi^\varepsilon + (u \cdot \nabla \xi) * \rho_\varepsilon dt + \sum_k (\sigma_k \cdot \nabla \xi) * \rho_\varepsilon dW^k - \frac{1}{2} \operatorname{tr}[a D^2 \xi^\varepsilon] dt = 0, \quad (5.4)$$

which also reads

$$\begin{aligned} d\xi^\varepsilon &+ u \cdot \nabla \xi^\varepsilon dt + \sum_k \sigma_k \cdot \nabla \xi^\varepsilon dW^k - \frac{1}{2} \text{tr}[aD^2 \xi^\varepsilon] dt = [u \cdot \nabla, \rho_\varepsilon *] \xi dt \\ &+ \sum_k [\sigma_k \cdot \nabla, \rho_\varepsilon *] \xi dW^k. \end{aligned}$$

Now, by (5.4), since  $\xi^\varepsilon$  is adapted regular (together with  $(u \cdot \nabla \xi) * \rho_\varepsilon$ ,  $\sigma_k \cdot \nabla \xi) * \rho_\varepsilon$ ,  $aD^2 \xi^\varepsilon$ ), we can apply Itô-Kunita-Wentzell formula (see e.g. Theorem 8.3, page 188 of [19], with easy modifications for the case of an infinite number of  $k$ 's), obtaining for  $\xi_t^\varepsilon(\Phi_t)$

$$d\xi_t^\varepsilon(\Phi_t) = [u_t \cdot \nabla, \rho_\varepsilon *] \xi_t(\Phi_t) dt + \sum_k [\sigma_k \cdot \nabla, \rho_\varepsilon *] \xi_t(\Phi_t) dW^k.$$

Since  $\Phi$  is measure-preserving, integrating in space we get

$$\begin{aligned} E \left[ \int_{\mathbb{T}^2} |\xi_t^\varepsilon(\Phi_t) - \xi_0| dx \right] &\leq \int_0^t \int_{\mathbb{T}^2} E[| [u_r \cdot \nabla, \rho_\varepsilon *] \xi_r |] dx dr \\ &+ \sum_k \int_0^t \int_{\mathbb{T}^2} E[| [\sigma_k \cdot \nabla, \rho_\varepsilon *] \xi_r |^2]^{1/2} dx dr. \end{aligned}$$

By the Commutator Lemma, for a.e.  $r$  and  $\omega$  in  $\Omega$ ,  $\int_{\mathbb{T}^2} |[u_r \cdot \nabla, \rho_\varepsilon *] \xi_r| dx$  tends to 0 as  $\varepsilon \rightarrow 0$ . Besides, this term is dominated by

$$C \|Du_r\|_{L^1(\mathbb{T}^2)} \| \xi_r \|_{L^\infty(\mathbb{T}^2)} \leq C' \|\xi\|_{L^\infty([0,T] \times \mathbb{T}^2 \times \xi)}^2.$$

Indeed, for every  $v$  in  $L^\infty(\mathbb{T}^2)$  and every finite  $p \geq 1$ ,  $\|D(K * v)\|_{L^p(\mathbb{T}^2)} \leq C \|D^2(-\Delta)^{-1} v\|_{L^p(\mathbb{T}^2)} \leq C' \|v\|_{L^\infty(\mathbb{T}^2)}$ . So dominated convergence theorem gives that

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathbb{T}^2} E[| [u_r \cdot \nabla, \rho_\varepsilon *] \xi_r |] dx dr = 0.$$

Similarly, for every  $k$ , for a.e.  $r$  and  $\omega$  in  $\Omega$ ,  $\int_{\mathbb{T}^2} |[\sigma_k \cdot \nabla, \rho_\varepsilon *] \xi_r|^2 dx$  tends to 0 as  $\varepsilon \rightarrow 0$  and is dominated by

$$C \|D\sigma_k\|_{L^2(\mathbb{T}^2)}^2 \|\xi_r\|_{L^\infty(\mathbb{T}^2)}^2.$$

Since  $\sum_k \|D\sigma_k\|_{L^2(\mathbb{T}^2)}^2 \leq \sum_k \|D\sigma_k\|_{L^\infty(\mathbb{T}^2)}^2 < +\infty$ , then we have (again by dominated convergence theorem)

$$\lim_{\varepsilon \rightarrow 0} \sum_k \int_0^t \int_{\mathbb{T}^2} E[| [\sigma_k \cdot \nabla, \rho_\varepsilon *] \xi_r |^2] dx dr = 0.$$



Thus, for any fixed  $t > 0$ ,  $\xi_t^\varepsilon(\Phi_t)$  tends to  $\xi_0$  in  $L^1(\mathbb{T}^2 \times \Omega)$  as  $\varepsilon \rightarrow 0$ . Since  $\xi_t^\varepsilon$  converges to  $\xi_t$  in  $L^1(\mathbb{T}^2 \times \Omega)$  (the convergence in  $L^1(\mathbb{T}^2)$  being dominated by  $\|\xi\|_{L^\infty}$ ) and  $\Phi_t$  is measure-preserving,  $\xi_t^\varepsilon(\Phi_t)$  converges to  $\xi_t(\Phi_t)$  in  $L^1(\mathbb{T}^2 \times \Omega)$  and thus  $\xi_t(\Phi_t) = \xi_0$ , which is our thesis.  $\square$

**Corollary 5.4.** *The uniqueness for the stochastic Euler vorticity equation (in the class of  $L^\infty$  solutions) holds.*

*Proof.* The above Proposition 5.3 tells that a solution  $\xi$  to stochastic Euler vorticity equation is completely determined by the associated flow  $\Phi$  which solves (4.1) with  $u = u^\xi$ ; again for the proposition,  $u = u^\Phi$  and so  $\Phi$  solves (2.3). Thus uniqueness for (2.3) implies uniqueness for stochastic Euler vorticity equation.  $\square$

This concludes the proof of Theorem 2.9.

## 6 Stability

In this section we want to prove a stability result, both at Lagrangian and Eulerian points of view, when the kernel  $K$  is regularized.

Precisely, take a family  $(\rho_\varepsilon)_\varepsilon$  of even compactly supported resolutions of identity and define  $K^\varepsilon := K * \rho_\varepsilon$ . Consider the approximated non-local ODE

$$\Phi_t^\varepsilon(x) = x + \int_0^t \int_{\mathbb{T}^2} K^\varepsilon(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y)) \xi_0(y) dy + \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^2} \sigma_k(\Phi_r^\varepsilon(x)) dW_r^k \quad (6.1)$$

and the approximated stochastic Euler vorticity equation

$$d\xi^\varepsilon + u^{\varepsilon, \xi^\varepsilon} \cdot \nabla \xi^\varepsilon dt + \sum_k \sigma_k \cdot \nabla \xi^\varepsilon dW^k = \frac{1}{2} C \Delta \xi^\varepsilon, \quad (6.2)$$

where  $u^{\varepsilon, \xi^\varepsilon} := K^\varepsilon * \xi^\varepsilon$ .

One can repeat all the previous definitions and arguments with  $K^\varepsilon$  in place of  $K$ , to get the analogues of Theorem 2.12 and Theorem 2.9: there exists a unique measure-preserving stochastic continuous flow  $\Phi$  solving (6.1), which is also  $C^\alpha$  in space, for every  $\alpha < 1$  and  $C^\beta$  in time, for every  $\beta < 1/2$ ; there exists a unique  $L^\infty$  distributional solution  $\xi^\varepsilon$  for (6.2). Moreover it holds

$$\xi_t^\varepsilon = (\Phi_t^\varepsilon)_\# \xi_0. \quad (6.3)$$

The first stability result is for flows:

**Proposition 6.1.** *The family  $(\Phi^\varepsilon)_\varepsilon$  converges to  $\Phi$  (as  $\varepsilon \rightarrow 0$ ) in  $C([0, T]; L^1(\mathbb{T}^2 \times \Omega))$ .*

*Proof.* The fact that  $\Phi^\varepsilon$  and  $\Phi$  belong to  $C([0, T]; L^1(\mathbb{T}^2 \times \Omega))$  can be proved easily, using similar techniques to those below. For the convergence, call  $Z_t^\varepsilon(x) = \Phi_t^\varepsilon(x) - \Phi_t(x)$ . As in the proof of Lemma 4.8, we would like to apply Itô formula for  $|Z^\varepsilon|$ . Proceeding as in Lemma 2.3 (applying Itô formula to  $f_\delta(x) = (|x|^2 + \delta)^{1/2}$ ), we get

$$\begin{aligned} E|Z_t^\varepsilon(x)| &\leq \int_0^t \int_{\mathbb{T}^2} E |K^\varepsilon(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y)) - K(\Phi_r(x) - \Phi_r(y))| |\xi_0(y)| dy dr \\ &\quad + C \int_0^t E|Z_r^\varepsilon(x)| dr. \end{aligned}$$

Integrating this inequality in  $x$ , since  $\xi_0$  is bounded, we obtain

$$\begin{aligned} &\int_{\mathbb{T}^2} E|Z_t^\varepsilon(x)| dx \\ &\leq \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E |K^\varepsilon(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y)) - K(\Phi_r(x) - \Phi_r(y))| dx dy dr \\ &\quad + C \int_0^t \int_{\mathbb{T}^2} E|Z_r^\varepsilon(x)| dx dr \\ &\leq \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E |K^\varepsilon(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y)) - K(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y))| dx dy dr \\ &\quad + \|\xi_0\|_{L^\infty} \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E |K(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y)) - K(\Phi_r(x) - \Phi_r(y))| dx dy dr \\ &\quad + C \int_0^t \int_{\mathbb{T}^2} E|Z_r^\varepsilon(x)| dx dr. \end{aligned} \tag{6.4}$$

For the first integral of (6.4), we exploit the fact that  $\Phi^\varepsilon$  is measure-preserving, for every  $\varepsilon$ ; so we have

$$\begin{aligned} &\int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E |K^\varepsilon(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y)) - K(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y))| dx dy dr \\ &= \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E |K^\varepsilon(x - y) - K(x - y)| dx dy dr \\ &\leq CT \int_{\mathbb{T}^2} |K^\varepsilon(x') - K(x')| dx', \end{aligned}$$

where we have used, in the last passage, the change of variable  $x - y = x'$ ,  $x + y = y'$  (this implies a change of domain, but the  $L^1$  norm of  $K^\varepsilon(x') - K(x')$  on the new domain is comparable with that on the torus). For the second integral of (6.4), we exploit the log-Lipschitz property of  $K$  (estimate (2.6)) and get

$$\begin{aligned}
& \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E |K(\Phi_r^\varepsilon(x) - \Phi_r^\varepsilon(y)) - K(\Phi_r(x) - \Phi_r(y))| dx dy dr \\
& \leq C \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E \gamma(|Z_r^\varepsilon(x) - Z_r^\varepsilon(y)|) dx dy dr \\
& \leq C \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} E [\gamma(|Z_r^\varepsilon(x)|) + \gamma(|Z_r^\varepsilon(y)|)] dx dy dr \\
& \leq C \int_0^t \int_{\mathbb{T}^2} \gamma(E|Z_r^\varepsilon(x)|) dx dr,
\end{aligned}$$

where we have used the sub-additivity of  $\gamma$  ( $\gamma(|x + y|) \leq \gamma(|x|) + \gamma(|y|)$ , as it can be easily checked) and Jensen inequality. Putting all together, we have

$$\int_{\mathbb{T}^2} E|Z_t^\varepsilon(x)| dx \leq C \|K^\varepsilon - K\|_{L^1(\mathbb{T}^2)} + C \int_0^t \int_{\mathbb{T}^2} \gamma(E|Z_r^\varepsilon(x)|) dx dr.$$

Again by comparison, we get  $\int_{\mathbb{T}^2} E|Z_t^\varepsilon(x)| dx \leq v(t, C \|K^\varepsilon - K\|_{L^1(\mathbb{T}^2)})$ , where  $v$  is defined as in (4.4). Since  $K$  is in  $L^1(\mathbb{T}^2)$ ,  $\|K^\varepsilon - K\|_{L^1(\mathbb{T}^2)}$  tends to 0 (as  $\varepsilon \rightarrow 0$ ), so

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^2} E|Z_t^\varepsilon(x)| dx \leq \sup_{t \in [0, T]} v(t, C \|K^\varepsilon - K\|_{L^1(\mathbb{T}^2)}) \rightarrow 0.$$

The proof is complete. □

Here is the result for the vorticity:

**Proposition 6.2.** *The family  $(\xi^\varepsilon)_\varepsilon$  converges weakly to  $\xi$  (as  $\varepsilon \rightarrow 0$ ), in the following sense: for every  $\varphi$  in  $C_b(\mathbb{T}^2)$ ,*

$$E \left| \int_{\mathbb{T}^2} \varphi \xi_t^\varepsilon dx - \int_{\mathbb{T}^2} \varphi \xi_t dx \right| \rightarrow 0$$

for every  $t$  and in  $L^p([0, T])$ , for any finite  $p$ .

*Proof.* First, notice that, by (6.3),

$$\int_{\mathbb{T}^2} \varphi \xi_t^\varepsilon dx = \int_{\mathbb{T}^2} \varphi(\Phi_t^\varepsilon) \xi_0 dx$$

and the same without  $\varepsilon$ . In particular,  $\varphi(\Phi_t^\varepsilon) \xi_0$  is dominated a.e. by a constant. Now fix the time  $t$ . We use here a classical argument in measure theory. Suppose by contradiction that there exist  $\delta > 0$  and a sequence  $\varepsilon_n \rightarrow 0$  such that

$$E \left| \int_{\mathbb{T}^2} \varphi(\Phi_t^{\varepsilon_n}) \xi_0 dx - \int_{\mathbb{T}^2} \varphi(\Phi_t) \xi_0 dx \right| \geq \delta. \quad (6.5)$$

The previous proposition gives that  $\Phi_t^{\varepsilon_n}$  converges to  $\Phi_t$  in  $L^1(\mathbb{T}^2 \times \Omega)$ . So we have for a subsequence  $\varepsilon_{n_k}$  that  $\Phi_t^{\varepsilon_{n_k}}$  tends to  $\Phi_t$  for a.e.  $(x, \omega)$  and similarly for  $\varphi(\Phi_t^{\varepsilon_{n_k}})$ , since  $\varphi$  is continuous. Hence, by dominated convergence theorem, we get that

$$E \left| \int_{\mathbb{T}^2} \varphi(\Phi_t^{\varepsilon_{n_k}}) \xi_0 dx - \int_{\mathbb{T}^2} \varphi(\Phi_t) \xi_0 dx \right| \rightarrow 0,$$

which contradicts (6.5). We have proved convergence at  $t$  fixed. Convergence in  $L^p([0, T])$ , for any finite  $p$ , follows from this result and dominated convergence theorem.  $\square$

## 7 An alternative way: reduction to the deterministic case

In this section we will see how to deduce the results in the stochastic case by a suitable transformation, assuming the deterministic case and more regularity for the  $\sigma_k$ 's. As we already said, we will not develop this method in all the details.

At a Lagrangian level (trajectories), consider the SDE with only the stochastic integral, namely

$$d\psi = \sum_k \sigma_k(\psi) \circ dW^k. \quad (7.1)$$

It is well known that, if the fields  $\sigma_k$ 's are regular enough ( $C^3$  should be sufficient,  $C^2$  is assumed in every “classical” result) and divergence-free, then there

exists a stochastic flows  $\psi$  of  $C^{1,1}(\mathbb{T}^2)$  measure-preserving diffeomorphisms solving (7.1) (a  $C^{1,1}$  diffeomorphism is a  $C^1$  map with Lipschitz-continuous derivatives, together with its inverse). The inverse flow  $\psi_t^{-1}$  satisfies

$$d\psi_t^{-1}(x) = - \sum_k \sigma_k(x) \cdot \nabla \psi_t^{-1}(x) \circ dW^k.$$

Now let  $\Phi$  be the Euler stochastic flow (solving (2.3)) and make a change of variable, composing with  $\psi_t^{-1}$ : call

$$\tilde{\Phi}(t, x, \omega) = \psi_{t,\omega}^{-1}(\Phi_{t,\omega}(x)). \quad (7.2)$$

Using the Itô-Kunita-Wentzell formula, we obtain the following random ODE for  $\tilde{\Phi}$ :

$$d\tilde{\Phi}_t = (D\psi_t)^{-1} u_t^\Phi(\psi_t(\tilde{\Phi}_t)) dt,$$

where  $u^\Phi$  is as in (2.4). This equation reads also as

$$d\tilde{\Phi}_t = \tilde{u}_t^{\tilde{\Phi}}(\tilde{\Phi}_t) dt, \quad (7.3)$$

where

$$\tilde{u}^{\tilde{\Phi}}(t, x, \omega) = (D\psi_{t,\omega}(x))^{-1} \int_{\mathbb{T}^2} K(\psi_{t,\omega}(x) - \psi_{t,\omega}(\tilde{\Phi}_{t,\omega}(y))) \xi_0(y) dy.$$

The equation (7.3) is not (3.1), but the drift  $\tilde{u}^{\tilde{\Phi}}$  has the same regularity properties of the drift  $u^\Phi$  of (3.1), provided  $\psi$  is a flow of  $C^{1,1}(\mathbb{T}^2)$  diffeomorphisms, since the term  $D\psi_t$  appears; here we need  $\sigma$  to be at least  $C^2$ . Thus, one could proceed as follows:

1. first we can repeat the argument in the deterministic part, to get existence and uniqueness for  $\tilde{\Phi}$  satisfying (7.3); since  $\psi$  is a regular flow adapted to the Brownian filtration, this implies strong existence and strong uniqueness for  $\Phi$  itself (plus homeomorphism property), i.e. Theorem 2.12;
2. then Section 5 applies and we deduce Theorem 2.9.

This can be seen also at an Eulerian level (velocity field). Heuristically, with the change of variable (7.2), we should consider, as new vorticity,  $\tilde{\xi}_t =$

$\xi_0(\tilde{\Phi}_t^{-1}) = \xi_t(\psi_t)$ . Indeed, let  $\xi$  be a solution to (1.1) and let  $\psi$  be as above, call

$$\tilde{\xi}(t, x, \omega) = \xi(t, \psi(t, x, \omega), \omega).$$

Applying, this time formally, the Itô-Kunita-Wentzell formula, we obtain the following random PDE for  $\tilde{\xi}$ :

$$\partial_t \tilde{\xi} + \tilde{u}^{\tilde{\xi}} \cdot \nabla \tilde{\xi} = 0, \quad (7.4)$$

where

$$\tilde{u}^{\tilde{\xi}} = (D\psi_{t,\omega}(x))^{-1} \int_{\mathbb{T}^2} K(\psi_{t,\omega}(x) - \psi_{t,\omega}(y)) \tilde{\xi}_t(y) dy.$$

This fact, as well as its converse (the passage from  $\tilde{\xi}$  to  $\xi$ ), can be made rigorous in the following way. First, we take  $\xi^\varepsilon = \xi * \rho_\varepsilon$  (where  $\rho_\varepsilon$  are even compactly supported mollifiers) and write the equation for  $\xi^\varepsilon$  (using commutators only for the  $\sigma_k$ 's):

$$\partial_t \xi^\varepsilon + (u^\xi \cdot \nabla \xi)^\varepsilon + \sum_k \sigma_k \cdot \nabla \xi^\varepsilon \circ \dot{W}^k - \sum_k [\sigma_k \cdot \nabla, \rho_\varepsilon *] \xi \circ \dot{W}^k = 0.$$

Then we multiply this equation by  $\varphi(\psi^{-1})$ , where  $\varphi$  is any regular test function on  $\mathbb{T}^2$ . In this way we obtain (7.4) for  $\xi^\varepsilon(\psi)$ , with  $(u^\xi \cdot \nabla \xi)^\varepsilon(\psi)$  in place of  $\tilde{u}^{\tilde{\xi}} \cdot \nabla \tilde{\xi}$  and with the additional commutator term  $\sum_k [\sigma_k \cdot \nabla, \rho_\varepsilon *] \xi(\psi) \circ \dot{W}^k$ . Finally we let  $\varepsilon$  go to 0, getting (7.4).

Again (7.4) is not the deterministic Euler vorticity equation ((1.1) with  $\sigma = 0$ ), but its drift  $\tilde{u}^{\tilde{\xi}}$  has the same regularity properties of the drift  $u^\xi$  of (3.1), provided  $\psi$  is a flow  $C^{1,1}(\mathbb{T}^2)$  diffeomorphisms. So one can repeat the arguments in the deterministic case (flows and commutator lemma), to get existence and uniqueness for the random PDE (7.4), then strong existence and strong uniqueness for 1.1 follow immediately.

Finally we mention that the passage between  $\xi$  and  $\tilde{\xi}$  can be seen at a more abstract level; this is a classical remark, due at least to Lamperti, Doss and Sussmann ([21], [12], [27]). Suppose to have an SPDE of the form

$$d\xi + A(\xi)\xi dt + \sum_k B_k \xi \circ dW^k = 0,$$

where  $A(x)$  and  $B_k$  are linear operators (for simplicity assume  $B_k$  time-independent); in our case,  $A(\xi) = u^\xi \cdot \nabla$  and  $B_k = \sigma_k \cdot \nabla$ . Consider formally

$$\tilde{\xi}_t = e^{\sum_k B_k W_t^k} \xi_t;$$

in our case, this corresponds to the composition  $\xi(\psi)$ . Then formally  $\tilde{\xi}$  satisfies the following random PDE:

$$\partial_t \tilde{\xi} + e^{\sum_k B_k W_t^k} A(e^{-\sum_k B_k W_t^k} \tilde{\xi}) e^{-\sum_k B_k W_t^k} \tilde{\xi} = 0.$$

Thus we have reduced an SPDE to a random PDE, which can be treated through deterministic techniques.

## A A useful inequality

This section contains a proof of an auxiliary inequality used in a crucial way twice in our paper.

**Lemma A.1.** *Assume that  $A, B > 0$  and  $T > 0$ . Suppose that  $(\rho_n)_{n=0}^\infty$  is a sequence of continuous nonnegative functions defined on the interval  $[0, T]$  such that for every  $\varepsilon \in (0, 1)$  and every  $n$ ,*

$$\rho_t^n \leq A \log \frac{1}{\varepsilon} \int_0^t \rho_s^{n-1} ds + \varepsilon B t, \quad t \in [0, T]. \quad (\text{A.1})$$

Then

$$\rho_t^n \leq \frac{(At)^n}{\sqrt{2\pi n}} \sup_{s \in [0, t]} |\rho_s^0| + Bt(e^{At-1})^n, \quad t \in [0, T].$$

*Proof of Lemma A.1.* By Induction one can show that for every  $n \in \mathbb{N}^*$  and every  $\varepsilon \in (0, 1)$ , with  $L_\varepsilon = \log \frac{1}{\varepsilon}$ ,

$$\begin{aligned} \rho_t^n &\leq (AL_\varepsilon)^n \int_0^t \dots \int_0^{s_2} \rho_{s_1}^0 ds_1 \dots ds_n \\ &+ B\varepsilon t \sum_{k=0}^{n-1} (AL_\varepsilon t)^k \int_0^t \dots \int_0^{s_2} ds_1 \dots ds_k \\ &\leq \frac{(AL_\varepsilon t)^n}{n!} \sup_{s \in [0, t]} |\rho_s^0| + B\varepsilon t \sum_{k=0}^{n-1} \frac{(AL_\varepsilon t)^k}{k!}, \quad t \in [0, T]. \end{aligned} \quad (\text{A.2})$$

Let us take  $n \in \mathbb{N}$ . Choose  $\varepsilon = e^{-n}$ . Then by the above inequality and

Stirling's inequality,

$$\begin{aligned}
\rho_t^n &\leq \frac{(Ant)^n}{n!} \sup_{s \in [0, t]} |\rho_s^0| + Be^{-nt} \sum_{k=0}^{n-1} \frac{(Ant)^k}{k!} \\
&\leq \frac{(Ant)^n}{n!} \sup_{s \in [0, t]} |\rho_s^0| + Bt(e^{At-1})^n \\
&\leq \frac{(At)^n}{\sqrt{2\pi n}} \sup_{s \in [0, t]} |\rho_s^0| + Bt(e^{At-1})^n, \quad t \in [0, T]. \tag{A.3}
\end{aligned}$$

This concludes the proof.  $\square$

**Corollary A.2.** *In the framework of the above Lemmma, if  $AT^* < 1$ , then  $\sup_{t \in [0, T^*]} \rho_t^n \rightarrow 0$ .*

*Proof.* If  $AT^* < 1$ , then  $\sup_{t \in [0, T^*]} \rho_t^n$  is bounded from above by a sum of the  $n$ -th terms of two convergent geometrical series.  $\square$

## B Proof of inequality (2.6)

We give a sketch of the proof of inequality (2.6). Call  $G$  is the Green function of the Laplacian on the torus  $\mathbb{T}^2 = [-1/2, 1/2]^2$  (with periodic boundary condition). We will prove:

**Proposition B.1.** *The function  $G$  is in  $C^\infty(\mathbb{T}^2 \setminus \{0\})$ . Its behaviour in 0 is given by*

$$|G(x)| \leq C(-\log |x| + 1)$$

*and that of its derivative  $D^{(n)}$ ,  $n$  positive integer, by*

$$|D^n G(x)| \leq C_n(|x|^{-n} + 1).$$

Assuming this result, we get that  $|K(x)| \leq C_1(|x|^{-1} + 1)$ . This implies the estimate (2.6) by an elementary argument (see [22], Appendix 2.3).

Proposition B.1 is a special case (at least for  $n \leq 2$ ) of a general fact, valid for compact  $C^\infty$  Riemannian manifolds of finite dimensions, see [3, section 4.2], for the statement and a proof. We give here a different proof, taken in spirit from [4] (which studies the 3D case).



*Sketch of the proof.* It is easy to see that the Fourier expansion of  $G$  is

$$G(x) = -\frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{1}{|k|^2} e^{2\pi i k \cdot x}$$

Since this expression seems not helpful in the analysis of regularity around 0, we will use the solution  $v$ , in  $L^2([0, T] \times \mathbb{T}^2)$ , of the heat equation

$$\partial_t v = \Delta v,$$

with initial condition  $v_0 = \delta_0 - 1$  (more precisely,  $v_t \rightharpoonup \delta_0 - 1$  as  $t \rightarrow 0$ ). It is easy to see that this unique solution can be expressed in two ways: one with its Fourier expansion, which is

$$v(t, x) = \sum_{k \in \mathbb{Z}^2, k \neq 0} e^{-4\pi^2 |k|^2 t} e^{2\pi i k \cdot x}, \quad (\text{B.1})$$

the other with Gaussian densities, that is

$$v(t, x) = -1 + \frac{1}{4\pi t} \sum_{l \in \mathbb{Z}^2} \exp \frac{-|x - l|^2}{4t}. \quad (\text{B.2})$$

One verifies, e.g. using (B.1), that

$$\begin{aligned} G(x) &= -\int_0^{+\infty} v(t, x) dt = -\int_1^{+\infty} v(t, x) dt - \int_0^1 v(t, x) dt \\ &=: -G_1(x) - G_2(x). \end{aligned}$$

Now  $G_1$  is in  $C^\infty(\mathbb{T}^2)$ , as one can see from its Fourier expansion, again from (B.1). For  $G_2$  we exploit (B.2):

$$\begin{aligned} G_2(x) &= \left( -1 + \int_0^1 \frac{1}{4\pi t} \sum_{l \in \mathbb{Z}^2, l \neq 0} \exp \frac{-|x - l|^2}{4t} dt \right) \\ &+ \int_0^1 \frac{1}{4\pi t} \exp \frac{-|x|^2}{4t} dt =: G_3(x) + G_4(x), \end{aligned}$$

the sum being between functions on  $\mathbb{R}^2$  (though  $x$  is still in  $[-1/2, 1/2]^2$ ). The first addend  $G_3$  is  $C^\infty$  on an open neighborhood of  $[-1/2, 1/2]^2$  (e.g.

$] - 3/4, 3/4[^2]$ : indeed, for any  $n$  nonnegative integer, we have

$$\begin{aligned}
& \int_0^1 \left| D^{(n)} \frac{1}{4\pi t} \sum_{l \in \mathbb{Z}^2, l \neq 0} \exp \frac{-|x-l|^2}{4t} \right| dt \\
& \lesssim \int_0^1 t^{-(2n+1)} \sum_{l \neq 0} \exp \frac{-|x-l|^2}{4t} dt \\
& \lesssim \int_0^1 t^{-(2n+1)} \sum_{h=1}^{\infty} \exp \frac{-h}{ct} dt \\
& \sim \int_0^1 t^{-(2n+1)} e^{-1/(ct)} dt < +\infty,
\end{aligned}$$

for some  $c > 0$  independent of  $x$ , when  $x$  is in  $] - 3/4, 3/4[^2$ . The second addend  $G_4$  is in  $C^\infty([ - 3/4, 3/4[^2 \setminus \{0\}))$ . So  $G$  is in  $C^\infty(\mathbb{T}^2 \setminus \{0\})$ . For the behaviour in 0, this is given by the behaviour of  $G_4$ , which is computed by standard techniques. We have, with the change of variable  $s = |x|^{-1/2}t$ ,

$$G_4(x) \sim \int_0^{|x|^{-1/2}} s^{-1} e^{-1/(4s)} ds \sim -\log |x|$$

and, for  $n \geq 1$ ,

$$|D^{(n)} G_4(x)| \sim |x|^{-n} \int_0^{|x|^{-1/2}} s^{-(2n+1)} e^{-1/(4s)} ds \sim |x|^{-n}.$$

The proof is complete. □

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